# Space-Time Foam Differential Algebras of Generalized Functions and a Global Cauchy-Kovalevskaia Theorem

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February 4, 2008

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**Abstract**

The new *global* version of the Cauchy-Kovalevskaia theorem presented here is a strengthening and extension of the *regularity* of similar global solutions obtained earlier by the author. Recently the space-time foam differential algebras of generalized functions with *dense* singularities were introduced. A main motivation for these algebras comes from the so called space-time foam structures in General Relativity, where the set of singularities can be dense. A variety of applications of these algebras have been presented elsewhere, including in de Rham cohomology, Abstract Differential Geometry, Quantum Gravity, etc. Here a global Cauchy-Kovalevskaia theorem is presented for arbitrary analytic nonlinear systems of PDEs. The respective global generalized solutions are analytic on the whole of the domain of the equations considered, except for singularity sets which are closed and nowhere dense, and which upon convenience can be chosen to have zero Lebesgue measure.

In view of the severe limitations due to the polynomial type growth conditions in the definition of Colombeau algebras, the class of singularities such algebras can deal with is considerably limited. Consequently, in such algebras one cannot even formulate, let alone obtain, the global version of the Cauchy-Kovalevskaia theorem presented in this paper.

"We do not possess any method at all to derive systematically solutions that are free of singularities..."

Albert Einstein
*The Meaning of Relativity*
Princeton Univ. Press, 1956, p. 165

# 1 Algebras of Generalized functions with Dense Singularities, or Space-Time Foam Algebras

## 1.1 Families of Dense Singularities in Euclidean Spaces

In this paper, following Rosinger [9-11,13,15], we consider differential algebras of generalized functions - called *space-time foam* algebras - which have significantly strengthened and extended properties with respect to the *singularities* they can deal with. Namely, this time the singularities can be arbitrary, including *dense* sets, and the only condition they have to satisfy is that their complementaries, that is, the set of nonsingular points, be also dense. This, among others, allows for singularity sets with a cardinal *larger* than that of the set of nonsingular points. For instance, in the case the domain is the real line, the set of singularities can be given by the uncountable set of all the irrational numbers, since its complementary, the set of rational numbers, is still dense, although it is only countable.

These space-time foam algebras are instances of the earlier nonlinear algebraic theory of generalized functions introduced and developed in Rosinger [1-8,13-15], Rosinger & Walus [1,2], Mallios & Rosinger [1], Mallios [1], see 46F30 at www.ams.org/msc/46Fxx.html
This general nonlinear algebraic theory has so far exhibited as particular cases a number of differential algebras of generalized functions, among them, the Colombeau algebras, see Grosser et.al. [p. 7].

The space-time foam algebras in this paper are able to deal with by far the *largest* class of singularities so far in the literature.
This fact proves to have useful *existence* and *regularity* consequences when it comes to the solutions in the global version of the Cauchy-Kovalevskaia theorem.

On the other hand, in view of the severe limitations due to the polynomial type growth conditions in the definition of Colombeau algebras, the class

of singularities such algebras can deal with is considerbaly limited. Consequently, in such algebras one cannot even formulate, let alone obtain, the global version of the Cauchy-Kovalevskaia theorem presented in this paper.

In this section, following Rosinger [9-11] where they were first introduced, we recall in short the construction of these new, namely, space-time foam algebras of generalized functions. For that purpose, first we have to introduce the *families of singularities* such algebras can deal with.

Let our underlying topological space $X$ be any nonvoid open subset of $\mathbb{R}^n$. The general case in the construction of space-time foam algebras, namely when $X$ is any finite dimensional smooth manifold, is presented in Rosinger [11], and rather surprisingly, it does not lead to any additional technical difficulties. This fact, in addition to the far larger class of singularities it can handle, is one of the advantages of the space-time foam algebras when compared, for instance, with the Colombeau algebras.

We shall consider various families of singularities in $X$, each such family being given by a corresponding set $\mathcal{S}$ of subsets $\Sigma \subset X$, with each such subset $\Sigma$ describing a possible set of singularities of a certain given generalized function, or in particular, generalized solution.

The *largest* family of singularities $\Sigma \subset X$ which we can consider is given by

$$\mathcal{S}_{\mathcal{D}} \;=\; \{ \; \Sigma \subset X \;\;|\;\; X \setminus \Sigma \text{ is dense in } X \; \} \tag{1.1}$$

In this way, the various families $\mathcal{S}$ of singularities $\Sigma \subset X$ which we shall deal with, will each satisfy the condition $\mathcal{S} \subseteq \mathcal{S}_{\mathcal{D}}$.

Regarding the treatment of *singularities* in a Differential Geometric context, it should be noted that a major interest in large, possibly *dense* sets of singularities comes from general relativity, see Finkelstein, Geroch [1,2], Heller [1-3], Heller & Sasin [1-3], Heller & Multarzynski & Sasin, Gurszczak & Heller, or Mallios [1-6], Mallios & Rosinger [1-3].

In this respect we note that, according to the strongest earlier corresponding result, see Heller [2], Heller & Sasin [2], the family of singularities $\mathcal{S}$ could only be composed from one single closed nowhere dense $\Sigma$, which in addition, had to be in the boundary of $X$.

On the other hand, in Mallios & Rosinger [1] - which except for Mallios & Rosinger [2,3], did treat the most general type of singularities - the family $\mathcal{S}$ could already contain *all closed and nowhere dense* subsets $\Sigma$ in $X$. And then finally, in Mallios & Rosinger [2,3] the *largest* class of singularities so far, namely such as in this paper, thus in particular, dense singularities as well, are treated.

For earlier developments regarding the possible treatment of singularities in a Differential Geometric context one can consult, for instance, Sikorski, Kirillov [1,2], Mostow, or Souriau [1,2]. And it should be mentioned that, as seen in Finkelstein, and especially in Geroch [1,2], the issue of singularities has for a longer time been of fundamental importance in General Relativity.

In this paper, as in Rosinger [9-11,13,15] and Mallios & Rosinger [2,3], the family $\mathcal{S}$ of singularities can be any subset of $\mathcal{S}_{\mathcal{D}}$ in (1.1). Among other ones, two such families which will be of interest are the following

$$\mathcal{S}_{nd} = \{ \Sigma \subset X \mid \Sigma \text{ is closed and nowhere dense in } X \} \tag{1.2}$$

and

$$\mathcal{S}_{Baire\ I} = \{ \Sigma \subset X \mid \Sigma \text{ is of first Baire category in } X \} \tag{1.3}$$

Obviously

$$\mathcal{S}_{nd} \subset \mathcal{S}_{Baire\ I} \subset \mathcal{S}_{\mathcal{D}} \tag{1.4}$$

## 1.2 Asymptotically Vanishing Ideals

Let us now for convenience recall shortly the idea of the construction introduced in Rosinger [9-11]. There are *two* basic ingredients involved. First, we take any family $\mathcal{S}$ of singularity sets $\Sigma \subset X$, family which satisfies the conditions

$$\begin{aligned} &\forall\ \Sigma \in \mathcal{S} : \\ &\quad X \setminus \Sigma \text{ is dense in } X \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\forall\ \Sigma, \Sigma' \in \mathcal{S} : \\ &\exists\ \Sigma'' \in \mathcal{S} : \\ &\quad \Sigma \cup \Sigma' \subseteq \Sigma'' \end{aligned} \tag{1.6}$$

Clearly, both families $\mathcal{S}_{nd}$ and $\mathcal{S}_{Baire\ I}$ satisfy conditions (1.5) and (1.6).

Now, as the second ingredient, and so far independently of $\mathcal{S}$ above, we take any right directed partial order $L = (\Lambda, \leq)$. In other words, $L$ is such that for each $\lambda, \lambda\,' \in \Lambda$, there exists $\lambda\,'' \in \Lambda$, with $\lambda, \lambda\,' \leq \lambda\,''$. The role of $L$ will become clear later, see for instance Example 1 in section 2.

Although we shall only be interested in singularity sets $\Sigma \in \mathcal{S}_{\mathcal{D}}$, the following ideal can be defined for any $\Sigma \subseteq X$. Indeed, let us denote by

$$\mathcal{J}_{L,\,\Sigma}(X) \tag{1.7}$$

the *ideal* in $(\mathcal{C}^\infty(X))^\Lambda$ of all the sequences of smooth functions indexed by $\lambda \in \Lambda$, namely, $w = (w_\lambda \,|\, \lambda \in \Lambda) \in (\mathcal{C}^\infty(X))^\Lambda$, sequences which *outside* of the singularity set $\Sigma$ will satisfy the *asymptotic vanishing* condition

$$\begin{array}{l} \forall\ x \in X \setminus \Sigma : \\ \exists\ \lambda \in \Lambda : \\ \forall\ \mu \in \Lambda,\ \mu \geq \lambda : \\ \forall\ p \in \mathbb{N}^n : \\ \quad D^p w_\mu(x) = 0 \end{array} \tag{1.8}$$

This means that the sequences of smooth functions $w = (w_\lambda \,|\, \lambda \in \Lambda)$ in the ideal $\mathcal{J}_{L,\,\Sigma}(X)$ may in a way *cover* with their support the singularity set $\Sigma$, and at the same time, they *vanish asymptotically* outside of it, together with all their partial derivatives.
In this way, the ideal $\mathcal{J}_{L,\,\Sigma}(X)$ carries in an *algebraic* manner the information on the singularity set $\Sigma$. Therefore, a *quotient* in which the factorization is made with such ideals may in certain ways *do away with* singularities, and do so through purely *algebraic* means, see (1.11), (1.12) below.

We note that the assumption about $L = (\Lambda, \leq)$ being right directed is used in proving that $\mathcal{J}_{L,\,\Sigma}(X)$ is indeed an ideal, more precisely that, for $w, w\,' \in \mathcal{J}_{L,\,\Sigma}(X)$, we have $w + w\,' \in \mathcal{J}_{L,\,\Sigma}(X)$.

Now, it is easy to see that for $\Sigma, \Sigma\,' \subseteq X$, we have

$$\Sigma \ \subseteq\ \Sigma\,' \quad \Longrightarrow \quad \mathcal{J}_{L,\,\Sigma}(X) \ \subseteq\ \mathcal{J}_{L,\,\Sigma\,'}(X) \tag{1.9}$$

in this way, in view of (1.6), it follows that

$$\mathcal{J}_{L,\mathcal{S}}(S) \;=\; \bigcup_{\Sigma\in\mathcal{S}} \mathcal{J}_{L,\Sigma}(X) \tag{1.10}$$

is also an ideal in $(\mathcal{C}^\infty(X))^\Lambda$.

### 1.3 Foam Algebras

In view of the above, for $\Sigma \subseteq X$, we can define the algebra

$$B_{L,\Sigma}(X) \;=\; (\mathcal{C}^\infty(X))^\Lambda / \mathcal{J}_{L,\Sigma}(X) \tag{1.11}$$

However, we shall only be interested in singularity sets $\Sigma \in \mathcal{S}_\mathcal{D}$, that is, for which $X \setminus \Sigma$ is *dense* in $X$. And in such a case the corresponding algebra $B_{L,\Sigma}(X)$ will be called a *foam algebra*.

### 1.4 Multi-Foam Algebras

With the given family $\mathcal{S}$ of singularities, and based on (1.10), we can now associate the *multi-foam algebra*

$$B_{L,\mathcal{S}}(X) \;=\; (\mathcal{C}^\infty(X))^\Lambda / \mathcal{J}_{L,\mathcal{S}}(X) \tag{1.12}$$

### 1.5 Space-Time Foam Algebras

The foam algebras and the multi-foam algebras introduced above will for the sake of simplicity be called together *space-time foam algebras*.

Clearly, if the family $\mathcal{S}$ of singularities consists of one single singularity set $\Sigma \in \mathcal{S}_\mathcal{D}$, that is, $\mathcal{S} = \{\,\Sigma\,\}$, then conditions (1.5), (1.6) are satisfied, and in this particular case the concepts of foam and multi-foam algebras are identical, in other words, $B_{L,\{\,\Sigma\,\}}(X) = B_{L,Sigma}(X)$. This means that the concept of multi-foam algebra is more general than that of foam algebra.

It is clear from their quotient construction that the space-time foam algebras are associative and commutative. However, the above constructions can easily be extended to the case when, instead of real valued smooth functions, we use smooth functions with values in an arbitrary *normed algebra*. In such a case the resulting space-time foam algebras will still be associative, but in general they may be noncommutative.

## 1.6 Space-Time Foam Algebras as Algebras of Generalized Functions

The reason why we restrict ourself to singularity sets $\Sigma \in \mathcal{S}_{\mathcal{D}}$, that is, to subsets $\Sigma \subset X$ for which $X \setminus \Sigma$ is dense in $X$, is due to the implication, see further details in Rosinger [15], and for a full argument Rosinger [4, chap. 3, pp. 65-119]

$$X \setminus \Sigma \text{ is dense in } X \quad \Longrightarrow \quad \mathcal{J}_{L,\,\Sigma}(X) \cap \mathcal{U}^\infty_\Lambda(X) \;=\; \{\,0\,\} \tag{1.13}$$

where $\mathcal{U}^\infty_\Lambda(X)$ denotes the *diagonal* of the power $(\mathcal{C}^\infty(X))^\Lambda$, namely, it is the set of all $u(\psi) = (\psi_\lambda \,|\, \lambda \in \Lambda)$, where $\psi_\lambda = \psi$, for $\lambda \in \Lambda$, while $\psi$ ranges over $\mathcal{C}^\infty(X)$. In this way, we have the algebra isomorphism $\mathcal{C}^\infty(X) \ni \psi \;\longmapsto\; u(\psi) \in \mathcal{U}^\infty_\Lambda(X)$.

This implication (1.13) follows immediately from the asymptotic vanishing condition (1.8). Indeed, if $\psi \in \mathcal{C}^\infty(X)$ and $u(\psi) \in \mathcal{J}_{L,\,\Sigma}(X)$, then (1.8) implies that $\psi = 0$ on $X \setminus \Sigma$, thus we must have $\psi = 0$ on $X$, since $X \setminus \Sigma$ was assumed to be dense in $X$. It follows, therefore, that the ideal $\mathcal{J}_{L,\,\Sigma}(X)$ is *off diagonal*.

The importance of (1.13) is that, for $\Sigma \in \mathcal{S}_{\mathcal{D}}$, it gives the following *algebra embedding* of the smooth functions into foam algebras

$$\mathcal{C}^\infty(X) \ni \psi \;\longmapsto\; u(\psi) + \mathcal{J}_{L,\,\Sigma}(X) \in B_{L,\,\Sigma}(X) \tag{1.14}$$

Now in view of (1.10), it is easy to see that (1.13) will as well yield the *off diagonality* property

$$\mathcal{J}_{L,\,\mathcal{S}}(X) \cap \mathcal{U}^\infty_\Lambda(X) \;=\; \{\,0\,\} \tag{1.15}$$

and thus similar with (1.14), we obtain the *algebra embedding* of smooth functions into multi-foam algebras

$$\mathcal{C}^\infty(X) \ni \psi \;\longmapsto\; u(\psi) + \mathcal{J}_{L,\,\mathcal{S}}(X) \in B_{L,\,\mathcal{S}}(X) \tag{1.16}$$

The algebra embeddings (1.14), (1.16) mean that the foam and multi-foam algebras are in fact *algebras of generalized functions*. Also they mean that the foam and multi-foam algebras are unital, with the respective unit elements $u(1) + \mathcal{J}_{L,\,\Sigma}(X)$, $u(1) + \mathcal{J}_{L,\,\mathcal{S}}(X)$.

Further, the asymptotic vanishing condition (1.8) also implies quite obviously that, for $\Sigma \subseteq X$, we have

$$D^p \mathcal{J}_{L,\Sigma}(X) \subseteq \mathcal{J}_{L,\Sigma}(X), \text{ for } p \in \mathbb{N}^n \tag{1.17}$$

where $D^p$ denotes the termwise $p$-th order partial derivation of sequences of smooth functions, applied to each such sequence in the ideal $\mathcal{J}_{L,\Sigma}(X)$.

Then again, in view of (1.10), we obtain

$$D^p \mathcal{J}_{L,\mathcal{S}}(X) \subseteq \mathcal{J}_{L,\mathcal{S}}(X), \text{ for } p \in \mathbb{N}^n \tag{1.18}$$

Now (1.17), (1.18) mean that the the foam and multi-foam algebras are in fact *differential algebras*, namely

$$D^p B_{L,\Sigma}(X) \subseteq B_{L,\Sigma}(X), \text{ for } p \in \mathbb{N}^n \tag{1.19}$$

where $\Sigma \in \mathcal{S}_{\mathcal{D}}$, and furthermore we also have

$$D^p B_{L,\mathcal{S}}(X) \subseteq B_{L,\mathcal{S}}(X), \text{ for } p \in \mathbb{N}^n \tag{1.20}$$

In this way we obtain that the foam and multi-foam algebras are *differential algebras of generalized functions*.

Also, the foam and multi-foam algebras contain the Schwartz distributions, that is, we have the *linear embeddings* which respect the arbitrary partial derivation of smooth functions

$$\mathcal{D}'(X) \subset B_{L,\Sigma}(X), \text{ for } \Sigma \in \mathcal{S}_{\mathcal{D}} \tag{1.21}$$

$$\mathcal{D}'(X) \subset B_{L,\mathcal{S}}(X) \tag{1.22}$$

Indeed, let us recall the wide ranging purely *algebraic characterization* of all those quotient type algebras of generalized functions in which one can embed linearly the Schwartz distributions, a characterization first given in 1980, see Rosinger [4, pp. 75-88], as well as Rosinger [5, pp. 306-315], Rosinger [6, pp. 234-244]. According to that characterization - which also contains the Colombeau algebras as a particular case - the *necessary and sufficient* condition for the existence of the linear embedding (1.21) is precisely the off diagonality condition in (1.13). Similarly, the necessary and sufficient condition for the existence of the linear embedding (1.22) is exactly the off diagonality condition (1.15).

One more property of the foam and multi-foam algebras will prove to be useful. Namely, in view of (1.10), it is clear that, for every $\Sigma \in \mathcal{S}$, we have

the inclusion $\mathcal{J}_{L,\Sigma}(X) \subseteq \mathcal{J}_{L,\mathcal{S}}$, and thus we obtain the *surjective algebra homomorphism*

$$B_{L,\Sigma}(X) \ni w + \mathcal{J}_{L,\Sigma}(X) \longmapsto w + \mathcal{J}_{L,\mathcal{S}}(X) \in B_{L,\mathcal{S}}(X) \tag{1.23}$$

And as we shall see in the next subsection, (1.23) can naturally be interpreted as meaning that the typical generalized functions in $B_{L,\mathcal{S}}(X)$ are *more regular* than those in $B_{L,\Sigma}X$.

## 1.7 Regularity of Generalized Functions

One natural way to interpret (1.23) in the given context of generalized functions is the following. Given two spaces of generalized functions $E$ and $F$, such as for instance

$$\mathcal{C}^\infty(X) \subset E \subset F \tag{1.24}$$

then the larger the space $F$ the *less regular* its typical element can appear to be, when compared with those of $E$. By the same token, the it smaller the space $E$ , the *more regular*, compared with those of $F$, one can consider its typical elements.

Similarly, given a *surjective* mapping

$$E \longrightarrow F \tag{1.25}$$

one can again consider that the typical elements of $F$ are at least as *regular* as those of $E$.

In this way, in view of (1.23), we can consider that, owing to the given *surjective* algebra homomorphism, the typical elements of the multi-foam algebra $B_{L,\mathcal{S}}(X)$ can be seen as being *more regular* than the typical elements of the foam algebra $B_{L,\Sigma}(X)$.
Furthermore, the algebra $B_{L,\mathcal{S}}(X)$ is obtained by factoring the same $(\mathcal{C}^\infty(X))^\Lambda$ as in the case of the algebra $B_{L,\Sigma}(X)$, this time however by the significantly *larger* ideal $\mathcal{J}_{L,\mathcal{S}_L}(X)$, an ideal which, unlike any of the individual ideals $\mathcal{J}_{L,\Sigma}(X)$, can simultaneously deal with *all* the singularity sets $\Sigma \in \mathcal{S}_L$, some, or in fact, many of which can be *dense* in $X$. Further details related to the connection between *regularization* in the above sense, and on the other hand, properties of *stability*, *generality* and *exactness* of generalized functions and solutions can be found in Rosinger [4-6].

This kind of interpretation will be used in section 3 related to the global Cauchy-Kovalevskaia theorem. Also, it will be further illustrated with the examples of the differential algebras of generalized functions presented in section 2.

# 2 On the Structure of Space-Time Foam Algebras

## 2.1 Special Families of Singularities

Since in section 3 the space-time foam algebras will be used in order to obtain global generalized solutions under the usual conditions of the Cauchy-Kovalevskaia theorem, it is useful to understand the structure of these algebras. And for that, one has to understand the structure of the ideals $\mathcal{J}_{L,\Sigma}(X)$ and $\mathcal{J}_{L,\mathcal{S}}(X)$. In particular, one has to have an idea about their *size*. Indeed, in view of the interpretation at the end of the previous subsection, the *larger* such ideals, the *more regular* the typical generalized functions in the corresponding quotient algebras (1.11), (1.12).

In order to be able gain more information about the mentioned algebras, we shall study a more particular case of them. This case is constructed by allowing a certain relationship between the right directed partially ordered sets $L = (\Lambda, \leq)$, and the families $\mathcal{S}$ of singularities. Namely, let us take associated with $L$ any set $\mathcal{S}_L$ of subsets $\Sigma$ of $X$, a set which satisfies the following three conditions. First

$$X \setminus \Sigma \text{ is dense in } X, \ \text{ for } \Sigma \in \mathcal{S}_L \tag{2.1}$$

then, second

$$\begin{array}{ll} \forall & \Sigma, \Sigma' \in \mathcal{S}_L : \\ \exists & \Sigma'' \in \mathcal{S}_L : \\ & \quad \Sigma \cup \Sigma' \ \subseteq \ \Sigma'' \end{array} \tag{2.2}$$

and finally, every $\Sigma \in \mathcal{S}_L$ can be represented as

$$\Sigma \ = \ \limsup_{\lambda \in \Lambda} \Sigma_\lambda \ = \ \bigcap_{\lambda \in \Lambda} \ \ \bigcup_{\mu \in \Lambda, \ \mu \geq \lambda} \Sigma_\mu \tag{2.3}$$

where $\Sigma_\lambda \subseteq X$, while $X \setminus \Sigma_\lambda$ is open, for $\lambda \in \Lambda$.

It is easy to see that we shall have $\mathcal{S}_L \subseteq \mathcal{S}_\mathcal{D}$, thus we are within the framework of the constructions in the previous section.

Further, let us assume that for two subsets $\Sigma, \Sigma' \subseteq X$ we have the representations $\Sigma = \limsup_{\lambda \in \Lambda} \Sigma_\lambda$ and $\Sigma' = \limsup_{\lambda \in \Lambda} \Sigma'_\lambda$, with $\Sigma_\lambda, \Sigma'_\lambda \subseteq X$, where $X \setminus \Sigma_\lambda, X \setminus \Sigma'_\lambda$, are open, for $\lambda \in \Lambda$. Then, for $\lambda \in \Lambda$, we define $\Sigma''_\lambda = \Sigma_\lambda \cup \Sigma'_\lambda$, hence $X \setminus \Sigma''_\lambda$ is open. In this way, in $X$, the subset $\Sigma'' = \limsup_{\lambda \in \Lambda} \Sigma''_\lambda$ has a representation (2.3), and clearly $\Sigma \cup \Sigma' \subseteq \Sigma''$.
This however, need not mean that (2.3) implies (2.2)) since $X \setminus \Sigma''$ need not be dense in $X$.

We also note that, for a suitable right directed partial order $L = (\Lambda, \leq)$, condition (2.3) is easy to satisfy for any nonvoid $\Sigma \subseteq X$. Indeed, let us take as $\Lambda$ the set of all $\lambda = A \subseteq \Sigma$, with nonvoid finite $A$. Further, for $\lambda = A$, $\mu = B \in \Lambda$, we define the right directed partial order relation $\lambda \leq \mu$ by the condition $A \subseteq B$. Finally, for $\lambda = A \in \Lambda$, we take $\Sigma_\lambda = A$, in which case relation (2.3) follows easily.

The above construction shows that in Euclidean spaces, it only has the following three different cases with respect to the size of $\Lambda$. First, when $\Sigma$ itself is finite. In this case the above construction can further be simplified, as we can take $\Lambda$ being composed of one element only, and with the trivial partial order on it, while we take $\Sigma_\lambda = \Sigma$. Then there are the two only other cases, when $\Sigma$ is countable, respectively, uncountable, and when correspondingly, $\Lambda$ can be taken countable or uncountable.
Obviously, we may expect to meet in various applications representations (2.3) which are more complicated than those constructed above, at least from the point of view of the partial orders $\leq$ on $\Lambda$, see for instance Example 1 below.

Let us note that $\mathcal{S}_{nd}$ in (1.2) satisfies (2.1) - (2.3), if in the last condition we take $L = (\Lambda, \leq) = \mathbb{N}$, while for a given $\Sigma$, and for each $\lambda = \nu \in \Lambda = \mathbb{N}$, we take $\Sigma_\lambda = \Sigma$.

Also, every $\Sigma \in \mathcal{S}_{Baire\ I}$, see (1.3), is of first Baire category, thus it is a countable union of nowhere dense sets in $X$. In this way $\mathcal{S}_{Baire\ I}$ satisfies (2.1) - (2.3) for the above $L$. We also note that the family of singularities $\mathcal{S}_{Baire\ I}$ contains plenty of singularity sets $\Sigma$ which are *dense* in $X$, and which in addition, have the cardinal of the continuum.

## 2.2 Special Ideals

Suppose now given a family $\mathcal{S}_L$ of singularities in $X$ satisfying (2.1) - (2.3). For any singularity set $\Sigma \in \mathcal{S}_L$ and any of its representations in (2.3), given by a particular family $\mathcal{S} = (\Sigma_\lambda \,|\, \lambda \in \Lambda)$, we denote by

$$\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X) \tag{2.4}$$

the *ideal* in $(\mathcal{C}^\infty(X))^\Lambda$ consisting of all the sequences of smooth functions indexed by the set $\Lambda$, namely, $w = (w_\lambda \,|\, \lambda \in \Lambda) \in (\mathcal{C}^\infty(X))^\Lambda$, sequences which *outside* of the singularity set $\Sigma$ will satisfy the *asymptotic vanishing* condition

$$\begin{array}{ll} \forall & x \in X \setminus \Sigma : \\ \exists & \lambda \in \Lambda : \\ \forall & \mu \in \Lambda,\ \mu \geq \lambda : \\ \exists & x \in \Delta_\mu \subseteq X \setminus \Sigma_\mu,\ \Delta_\mu \text{ open } : \\ & w_\mu = 0 \text{ on } \Delta_\mu \end{array} \tag{2.5}$$

In other words; the sequences of smooth functions $w = (w_\lambda \,|\, \lambda \in \Lambda)$ in the ideal $\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X)$ are in certain ways *covering* with their support the singularity set $\Sigma$, while outside of it, they are vanishing asymptotically.
Also, the ideal $\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X)$ carries in an *algebraic* manner the information on the singularity set $\Sigma$. It follows that a *quotient* in which the factorization is made with such ideals may in certain ways *do away with singularities* through purely algebraic means, see (2.11), (2.13) below.

The ideal $\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X)$ appears to depend not only on $L$ and $\Sigma$ but also on the family $S = (\Sigma_\lambda \,|\, \lambda \in \Lambda)$ which is in the particular representation of $\Sigma$ in (2.3). However, as we shall see next in Lemma 1, the ideal $\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X)$ only depends on $L$ and $\Sigma$, and does *not* depend on $S$, thus on the representation in (2.3). Therefore, from now on, this ideal will be denoted by

$$\mathcal{I}_{L,\,\Sigma}(X) \tag{2.6}$$

**Lemma 1**

The ideal $\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X)$ does *not* depend on $S = (\Sigma_\lambda \,|\, \lambda \in \Lambda)$ in the representation (2.3) of $\Sigma$.

**Proof.**

Let $S = (\Sigma_\lambda \,|\, \lambda \in \Lambda)$ and $S\,' = ({\Sigma_\lambda}' \,|\, \lambda \in \Lambda)$ be two representation (2.3) of $\Sigma$. We prove now that

$$\mathcal{I}_{L,\,\Sigma,\,\mathcal{S}}(X) \;\subseteq\; \mathcal{I}_{L,\,\Sigma,\,\mathcal{S}\,'}(X) \tag{2.7}$$

Let us take any sequence of smooth functions $w = (w_\lambda \,|\, \lambda \in \Lambda)$ in the left hand term of (2.7). Then (2.5) holds for $S$. In particular, for every given $x \in X \setminus \Sigma$, we can find $\lambda \in \Lambda$, such that for each $\mu \in \Lambda,\ \mu \geq \lambda$, there exists $x \in \Delta_\mu \subseteq X \setminus \Sigma$, $\Delta_\mu$ open, and $w_\mu = 0$ on $\Delta_\mu$.
We now show that by replacing $S$ with $S\,'$, we still have (2.5), for $x \in X \setminus \Sigma$ arbitrarily given as above. Indeed, in view of (2.3) corresponding to the representation of $\Sigma$ given now by $S\,'$, there exists $\lambda\,' \in \Lambda$ such that for all $\mu \in \Lambda,\ \mu \geq \lambda\,'$, we have $x \in X \setminus {\Sigma_\mu}'$, and $X \setminus {\Sigma_\mu}'$ is by assumption open.
Let us take $\lambda\,'' \in \Lambda$ with $\lambda\,'' \geq \lambda,\, \lambda\,'$. Then for all $\mu \in \Lambda,\ \mu \geq \lambda\,''$, we obviously have $x \in {\Delta_\mu}'' \subseteq X \setminus {\Sigma_\mu}'$, where ${\Delta_\mu}'' = \Delta_\mu \cap (X \setminus {\Sigma_\mu}')$ is open.
And clearly, $w_\mu = 0$ on ${\Delta_\mu}''$, since ${\Delta_\mu}'' \subseteq \Delta_\mu$.
In this way (2.5) does indeed hold for $S\,'$ as well, and the proof of (2.7) is completed.
Applying (2.7) the other way around, we obtain the proof of Lemma 1.

□

Now we establish a few properties of the ideals (2.6). It is easy to see that, for $\Sigma \in \mathcal{S}_L$, we have

$$\mathcal{I}_{L,\,\Sigma}(X) \;\subseteq\; \mathcal{J}_{L,\,\Sigma}(X) \tag{2.8}$$

also

$$D^p\,\mathcal{I}_{L,\,\Sigma}(X) \;\subseteq\; \mathcal{I}_{L,\,\Sigma}(X), \text{ for } p \in \mathbb{N}^n \tag{2.9}$$

as well as

$$\mathcal{I}_{L,\,\Sigma}(X) \cap \mathcal{U}^\infty_\Lambda(X) \;=\; \{\ 0\ \} \tag{2.10}$$

## 2.3 Special Foam Algebras

Let us consider the *special foam algebras*, Rosinger [9,10], which are defined as follows for every given $\Sigma \in \mathcal{S}_L$

$$A_{L,\,\Sigma}(X) \;=\; (\mathcal{C}^\infty(X))^\Lambda / \mathcal{I}_{L,\,\Sigma}(X) \tag{2.11}$$

In view of (2.9), (2.10), these are associative, commutative and unital differential algebras of generalized functions which contain the Schwartz distributions. And according to (2.8), for $\Sigma \in \mathcal{S}_L$ we have the *surjective algebra homomorphism*

$$A_{L,\Sigma}(X) \ni w + \mathcal{I}_{L,\Sigma}(X) \longmapsto w + \mathcal{J}_{L,\Sigma}(X) \in B_{L,\Sigma}(X) \tag{2.12}$$

And as also in the case of in (1.23), this can be interpreted as indicating that the typical generalized functions in $B_{L,\Sigma}(X)$ are *more regular* than the typical generalized functions in $A_{L,\Sigma}(X)$. More details related to this interpretation were presented in subsection 1.7.

## 2.4 Special Multi-foam Algebras

Now in order to deal simultaneously with all the singularity sets $\Sigma \in \mathcal{S}_L$, we define

$$\mathcal{I}_{L,\mathcal{S}_L}(X) = \bigcup_{\Sigma \in \mathcal{S}_L} \mathcal{I}_{L,\Sigma}(X) \tag{2.13}$$

which is the *ideal* in $(\mathcal{C}^\infty(X))^\Lambda$ generated by all ideals $\mathcal{I}_{L,\Sigma}(X)$, with $\Sigma \in \mathcal{S}_L$. Indeed, this is an ideal, since similar with (1.9), we have the implication in Lemma 2 below.

Finally, we can define the so called *special multi-foam algebra* of generalized functions, see Rosinger [9,10]

$$A_{L,\mathcal{S}_L}(X) = (\mathcal{C}^\infty(X))^\Lambda / \mathcal{I}_{L,\mathcal{S}_L}(X) \tag{2.14}$$

Together, both the special foam algebras (2.11) and the special multi-foam algebras in (2.14), will for simplicity be called *special space-time foam* algebras.

**Lemma 2.**

If $\Sigma,\ \Sigma\,' \in \mathcal{S}_L$, then

$$\Sigma \subseteq \Sigma\,' \implies \mathcal{I}_{L,\Sigma}(X) \subseteq \mathcal{I}_{L,\Sigma\,'}(X) \tag{2.15}$$

**Proof.**

Let $\Sigma, \Sigma\,'$ have the corresponding representations in (2.3) given by $S = (\Sigma_\lambda \mid \lambda \in \Lambda)$ and $S\,' = (\Sigma_\lambda\,' \mid \lambda \in \Lambda)$, respectively. Then in view of Lemma 1, the relation (2.5) will now hold for $\Sigma, S$, and any $w = (w_\lambda \mid \lambda \in \Lambda) \in \mathcal{I}_{L,\Sigma}$. We want to show that (2.5) also holds for $\Sigma\,', S\,'$ and $w$. Indeed, let $x \in$

$X \setminus \Sigma'$. Then $x \in X \setminus \Sigma$, since $\Sigma \subseteq \Sigma'$. Hence the assumption $w \in \mathcal{I}_{L,\Sigma}(X)$, together with (2.5) give $\lambda \in \Lambda$ such that for every $\mu \in \Lambda$, $\mu \geq \lambda$ there exists $x \in \Delta_\mu \subseteq X \setminus \Sigma_\mu$, with $w_\mu = 0$ on $\Delta_\mu$ and $\Delta_\mu$ is open.
On the other hand, in view of (2.3), we note that $X \setminus {\Sigma_\mu}'$ is open, for $\mu \in \Lambda$. Also $x \in X \setminus \Sigma'$ gives $\lambda' \in \Lambda$ such that $x \in X \setminus {\Sigma_\mu}'$, for every $\mu \in \Lambda$, $\mu \geq \lambda'$. Let us take $\lambda'' \in \Lambda$, $\lambda'' \geq \lambda, \lambda'$. It follows that for every $\mu \in \Lambda$, $\mu \geq \lambda''$ we shall have

$$x \in {\Delta_\mu}' \; = \; \Delta \cap (X \setminus {\Sigma_\mu}') \; \subseteq \; X \setminus {\Sigma_\mu}'$$

and clearly, by its above definition, ${\Delta_\mu}'$ is open. But $w_\mu = 0$ on ${\Delta_\mu}'$, as ${\Delta_\mu}' \subseteq \Delta_\mu$.
In this way, indeed, $w \in \mathcal{I}_{L,\Sigma'}(X)$, and thus the proof of (2.15) is completed.
□

Similar with (2.8) - (2.10), we have the properties

$$\begin{aligned} &\mathcal{I}_{L,\Sigma}(X) \;\subseteq\; \mathcal{J}_{L,\Sigma}(X) \;\subseteq\; \mathcal{J}_{L,\mathcal{S}_L}(X), \text{ for } \Sigma \in \mathcal{S}_L \\ &\mathcal{I}_{L,\Sigma}(X) \;\subseteq\; \mathcal{I}_{L,\mathcal{S}_L}(X) \;\subseteq\; \mathcal{J}_{L,\mathcal{S}_L}(X), \text{ for } \Sigma \in \mathcal{S}_L \end{aligned} \tag{2.16}$$

$$D^p \mathcal{I}_{L,\mathcal{S}_L}(X) \;\subseteq\; \mathcal{I}_{L,\mathcal{S}_L}(X), \text{ for } p \in \mathbb{N}^n \tag{2.17}$$

$$\mathcal{I}_{L,\mathcal{S}_L}(X) \cap \mathcal{U}^\infty_\Lambda(X) \; = \; \{0\} \tag{2.18}$$

Again, (2.17), (2.18) imply that the special multi-foam algebras $A_{L,\mathcal{S}_L}(X)$ are associative, commutative and unital differential algebras of generalized functions which contain the Schwartz distributions. Further, in view of (2.16), for $\Sigma \in \mathcal{S}_L$, we have the *surjective algebra homomorphism*

$$A_{L,\Sigma}(X) \ni w + \mathcal{I}_{L,\Sigma}(X) \;\; \longmapsto \;\; w + \mathcal{I}_{L,\mathcal{S}_L}(X) \in A_{L,\mathcal{S}_L}(X) \tag{2.19}$$

as well as the *surjective algebra homomorphism*

$$A_{L,\mathcal{S}_L}(X) \ni w + \mathcal{I}_{L,\mathcal{S}_L}(X) \;\; \longmapsto \;\; w + \mathcal{J}_{L,\mathcal{S}_L}(X) \in B_{L,\mathcal{S}_L}(X) \tag{2.20}$$

which together with (2.12), (1.23) will give the commutative diagram of *surjective* algebra homomorphisms

$$
\begin{tikzcd}
 & B_{L,\,\Sigma}(X) \arrow[dr] & \\
A_{L,\,\Sigma}(X) \arrow[ur] \arrow[dr] & & B_{L,\,\mathcal{S}_L}(X) \\
 & A_{L,\,\mathcal{S}_L}(X) \arrow[ur] &
\end{tikzcd}
\tag{2.21}
$$

In view of subsection 1.7., the interpretation of (2.21) is that the typical generalized function in the algebras which are the target of arrows are *more regular* than those in the algebras which are the source of the arrows. In particular the *most regular* differential algebras of generalized functions among those constructed in this paper are the *multi-foam algebras* $B_{L,\,\mathcal{S}_L}(X)$.

## 2.5 Special Space-Time Foam Algebras

Again, it will on occasion be convenient to call the special foam algebras and the special multi-foam algebras by one single term, namely, special space-time foam algebras.

As an illustration of the above, let us recall the *nowhere dense* differential algebras of generalized functions $A_{nd}(X)$ introduced in Rosinger [3], see also Rosinger [4-8], and section 3 below, which were recently used in Mallios & Rosinger [1] as the structure coefficients replacing the smooth functions in the abstract differential geometry developed in Mallios [1,2].
Namely,if we take as our right directed partial order the natural numbers $\mathbb{N}$, that is, $L = (\Lambda, \leq) = \mathbb{N}$, while we take the family of singularities in $X$ as given by $\mathcal{S}_{nd}$ in (1.2), then it is easy to see that

$$A_{nd}(X) \;=\; B_{\mathbb{N},\,\mathcal{S}_{nd}}(X) \;=\; A_{\mathbb{N},\,\mathcal{S}_{nd}}(X) \tag{2.22}$$

This, in particular, further clarifies the extent to which Mallios & Rosinger [2,3] strengthens the results in Mallios & Rosinger [1]. Indeed in the latter paper the singularity sets $\Sigma \subset X$ had to be closed and nowhere dense in $X$, thus their complementaries were open and dense in $X$. On the other hand, in Mallios & Rosinger [2,3], we only ask that for each singularity set $\Sigma \subset X$, the corresponding set of nonsingular points $X \setminus \Sigma$ be dense in $X$.

In this regard, the best previous results in the literature only allowed one single singularity set $\Sigma \subset X$, which in addition had to be closed and nowhere dense and in the boundary of $X$, see Heller [2], Heller & Sasin [2]. Consequently, already the result in Mallios & Rosinger [1] - which is considerably more particular than that in Mallios & Rosinger [2,3] - proved to be significantly more powerful, since it could deal simultaneously with *all* closed and nowhere dense singularity sets $\Sigma \subset X$, be they in the boundary of $X$ or not.

### 2.6 On the Structure of the Ideals

Let us show in Example l next, that for *large* enough - that is, uncountable and of the cardinality of the continuum - index sets $\Lambda$, the ideals $\mathcal{I}_{L,\Sigma}(X)$ are *not* trivial, namely, the do *not* collapse to the null ideal $\{ 0 \}$. For that purpose, for any singularity set $\Sigma \in \mathcal{S}_{\mathcal{D}}$, and for suitably chosen index sets $\Lambda$, we shall construct *large classes* of sequences of smooth functions

$$w^* \ = \ (w^*_\lambda \,|\, \lambda \in \Lambda) \in (\mathcal{C}^\infty(X))^\Lambda$$

such that $w^* \in \mathcal{I}_{L,\Sigma}(X)$. Then in view of (2.16), this will suffice to show that none of the other ideals $\mathcal{J}_{L,\Sigma}(X), \mathcal{J}_{L,\mathcal{S}}(X), \mathcal{I}_{L,\mathcal{S}_L}(X)$ is trivial.

The construction in Example 1 next will also illustrate in more detail the way *dense* singularities can be dealt with by purely *algebraic* means.

Here we should note that the issue of nontriviality of these ideals is in itself nontrivial. Indeed, in view of the off diagonality conditions (1.15), or equivalently, of the algebra embeddings (1.16), none of the ideals $\mathcal{J}_{L,\Sigma}(X), \mathcal{J}_{L,\mathcal{S}}(X)$, $\mathcal{I}_{L,\Sigma}(X)$ or $\mathcal{I}_{L,\mathcal{S}_L}(X)$ can be too large. Thus the issue of the nontriviality of these ideals involves a conflict.
As for determining which are the *maximal* ones among such ideals this is still an open problem, and one that has an obvious importance, as argued in some detailed in Rosinger [15].

**Example 1.**

Given any nonvoid singularity $\Sigma \subset X$ such that $\Sigma \in \mathcal{S}_{\mathcal{D}}$, in other words, for which $X \setminus \Sigma$ is dense in $X$, let us take the index set $\Lambda$ as the set of elements

$$\lambda \ = \ (A, (\alpha_x \,|\, x \in \Sigma))$$

where $A \subseteq \Sigma$ is nonvoid finite, and for $x \in \Sigma$, we have $\alpha_x \in \mathcal{D}(X)$, $\alpha_x \neq 0$. Here we recall that $\mathcal{D}(X)$ denotes the space of compactly supported smooth

functions on $X$.

Now we define on $\Lambda$ a *right directed* partial order $\leq$ as follows. Given $\lambda = (A, (\alpha_x \,|\, x \in \Sigma))$, $\mu = (B, (\beta_x \,|\, x \in \Sigma))$, we shall write $\lambda \leq \mu$, if and only if

$$A \subseteq B \quad \text{ and } \quad \bigcup_{x \in \Sigma} \text{supp } \beta_x \;\subseteq\; \bigcup_{x \in \Sigma} \text{supp } \alpha_x \tag{2.23}$$

Further, aiming to obtain for $\Sigma$ a representation (2.3) , for $\lambda = (A, (\alpha_x \,|\, x \in \Sigma)) \in \Lambda$, we define $\Sigma_\lambda = A$. Finally, we also define the compactly supported smooth function

$$w^*_\lambda \;=\; \sum_{x \in A} \alpha_x \in \mathcal{D}(X) \tag{2.24}$$

which is well defined since $A$ is a nonvoid finite set.

Then it follows easily that $X \setminus \Sigma_\lambda$ is open for $\lambda \in \Lambda$, and in addition we also have

$$\Sigma \;=\; \limsup_{\lambda \in \Lambda} \Sigma_\lambda \tag{2.25}$$

The fact of interest to us is that

$$w^* \;=\; (w^*_\lambda \,|\, \lambda \in \Lambda) \in \mathcal{I}_{L,\Sigma}(X) \tag{2.26}$$

Indeed, for the proof of (2.26), let us take any $y \in X \setminus \Sigma$, then (2.25) gives

$$\begin{array}{ll} \exists & \lambda = (A, (\alpha_x \,|\, x \in \Sigma)) \in \Lambda : \\ \forall & \mu = (B, (\beta_x \,|\, x \in \Sigma)) \in \Lambda, \; \mu \geq \lambda : \\ & y \in X \setminus \Sigma_\mu \;=\; X \setminus B \end{array}$$

But obviously, we can assume that

$$y \notin \bigcup_{x \in \Sigma} \text{ supp } \alpha_x \tag{2.27}$$

since we took $y \in X \setminus \Sigma$, while for $x \in \Sigma$, we have $\alpha_x \in \mathcal{D}(X)$, and the only other condition $\alpha_x$ has to satisfy is that $\alpha_x(x) \neq 0$.

Now let $\mu = (B, (\beta_x \,|\, x \in \Sigma)) \in \Lambda, \; \mu \geq \lambda$, then $A \subseteq B$ and

$$\bigcup_{x \in \Sigma} \text{supp } \beta_x \;\subseteq\; \bigcup_{x \in \Sigma} \text{supp } \alpha_x$$

hence the previous assumption (2.27) gives

$$y \notin \bigcup_{x \in \Sigma} \text{supp } \beta_x$$

thus

$$\exists\, \Delta_\mu \text{ open}, \;\; y \in \Delta_\mu \subseteq X \setminus \Sigma_\mu = X \setminus B \; :$$

$$\Delta_\mu \bigcap \; ( \bigcup_{x \in B} \text{ supp } \beta_x \; ) \;\; = \;\; \phi$$

In this way we obtain that $w^*_\mu = \sum_{x \in B} \beta_x = 0$ on $\Delta_\mu$, and in view of (2.5), the proof of (2.25) is completed.

□

From the point of view of dealing with *dense* singularities, the essential property in Example 1 above is illustrated in the sequences of smooth functions constructed in (2.26), namely

$$w^* \;\; = \;\; (w_\lambda` \,|\, \lambda \in \Lambda) \in \mathcal{I}_{L,\,\Sigma}(X) \tag{2.28}$$

Indeed, in view of (2.27), these sequence have the property

$$\begin{array}{ll} \forall & y \in X \setminus \Sigma : \\ \exists & \alpha_x \in \mathcal{D}(X), \text{ for each } x \in \Sigma : \\ & y \notin \bigcup_{x \in \Sigma} \text{ supp } \alpha_x \end{array} \tag{2.29}$$

which means that

$$\begin{array}{ll} \forall & x \in X \setminus \Sigma \; : \\ \exists & \lambda \in \Lambda \; : \\ \forall & \mu \in \Lambda, \;\; \mu \geq \lambda \; : \\ & \;\; x \notin \text{supp } w^*_\mu \end{array} \tag{2.30}$$

On the other hand, owing to the specific definition of $\Lambda$ in Example 1, it follows that

$$
\begin{aligned}
\forall \quad & \lambda = (A, (\alpha_x \,|\, x \in \Sigma)) \in \Lambda \; : \\
& \phi \neq \Sigma \;\subseteq\; \bigcup_{x \in \Sigma} \text{supp } \alpha_x
\end{aligned}
\tag{2.31}
$$

Here, as a consequence, we note *four* facts related to the sequences in (2.28).

- First, in (2.30) the *singularity* sets $\Sigma \subset X$ can be arbitrary large, provided that their complementary $X \setminus \Sigma$ are still dense in $X$. In particular, $\Sigma$ can have the cardinal of the continuum while $X \setminus \Sigma$ need only be countable and dense in $X$. As we mentioned, in the case of the real line $X = \mathbb{R}$, for instance, $\Sigma$ can be the uncountable set of all irrational numbers, since its complementary $X \setminus \Sigma$, that is, the rational numbers, is still dense in $X$, although it is only countable.
  And yet, every point $x$ outside of such rather arbitrary singularity sets $\Sigma$ will eventually also be outside of the support of $w^*_\lambda$, see earlier comment following (1.8).

- Second, due to (2.31), such rather arbitrary singularity sets $\Sigma \subset X$ will nevertheless be included in the support of the functions $\alpha_x$ which through (2.24), make up the terms of the sequences in (2.28), sequences which guarantee the nontriviality of the mentioned ideals.

- Third, the *index* set $\Lambda$ can *depend* on the given singularity set $\Sigma$, and can be rather *large*. In particular, it may even have to be uncountable and of the cardinality of the continuum, as happens in Example 1 above, and for the corresponding sequences in (2.28).

- Finally, we note that the above can give us a certain information about the possible *size* of the various deals we have considered so far. Indeed, in view of (2.16), we obtain for every singularity set $\Sigma \in \mathcal{S}_L$

$$
\begin{aligned}
& w^* \in \mathcal{I}_{L,\Sigma}(X) \;\subseteq\; \mathcal{J}_{L,\Sigma}(X) \;\subseteq\; \mathcal{J}_{L,\mathcal{S}_L}(X) \\
& w^* \in \mathcal{I}_{L,\Sigma}(X) \;\subseteq\; \mathcal{I}_{L,\mathcal{S}_L}(X) \;\subseteq\; \mathcal{J}_{L,\mathcal{S}_L}(X)
\end{aligned}
\tag{2.32}
$$

# 3 Global Cauchy-Kovalevskaia Theorem

## 3.1 Preliminary Comments

Let us recall that, as mentioned, in Rosinger [1-8] a large variety, and in fact, *infinitely* many classes of differential algebras of generalized functions were constructed.
And in case these constructions start with Banach algebra valued, and not merely with real or complex valued functions, the resulting algebras can be *noncommutative* as well. Also a wide ranging and purely algebraic characterization was given in Rosinger [4-6] for those algebras which contain the linear vector space of Schwartz distributions. That characterization is expressed by the *off diagonality* condition whose specific instance can be seen in (1.15), for instance. And in view of that characterization, the Colombeau algebras prove to be a particular case of the algebras constructed earlier in Rosinger [1-4], see details in Rosinger [5,6] or Grosser et.al. [p. 7].

Until recently, only two particular cases of these classes of algebras have been used in the study of global generalized solutions of nonlinear PDEs. Namely, first was the class of the nowhere dense differential algebras of generalized functions, see Rosinger [1-8], or (2.22) above, while later came the class of algebras considered in Colombeau.
These latter algebras, since they also contain the Schwartz distributions are - in view of the above mentioned algebraic characterization - by necessity a particular case of the classes of algebras of generalized functions first introduced in Rosinger [1-8].

The Colombeau algebras of generalized functions enjoy a rather simple and direct connection with the Schwartz distributions, and therefore, with a variety of Sobolev spaces as well. Furthermore, the polynomial type growth conditions which define - and also seriously limit with respect to the singularities which they are able to handle - the generalized functions in the Colombeau algebras, can offer an easy and familiar set up to work with for certain analysts. This led to their relative popularity in the study of generalized solutions of PDEs.
What happens, however, is that the ease and familiarity of working with growth type conditions not only restricts the class of singularities which can be dealt with, but also leads quite soon to considerable technical complications.
One such instance can be seen when comparing the difficulties in defining

on finite dimensional smooth manifolds the Colombeau algebras, see for instance, Grosser et.al., and on the other hand, the rather immediate and natural manner in which space-time foam algebras can be defined on the same manifolds, see Rosinger [11].

Another instance, related to the main subject of this paper, and already mentioned, is the following. In view of the severe limitations on the class of singularities the Colombeau algebras of generalized functions are able to deal with, one simply cannot formulate, let alone obtain in such algebras the global version of the Cauchy-Kovalevskaia theorem presented in this paper.

However, as mentioned in Rosinger [7, pp. 5-8, 11-12, 173-187], one should avoid rushing into a too early normative judgement about the way the long established *linear* theories of generalized functions - such as for instance the Schwartz or Sobolev distributions - should relate to the still emerging, and far more complex and rich corresponding *nonlinear* theories.
In particular, *two* aspects of such possible relationships still await a more thoroughly motivated and clear settlement :

- First, the purely algebraic-differential type connections between Schwartz distributions and the more recently constructed variety of differential algebras of generalized functions should be studied in more detail. And since the main aim is to deal with generalized - hence, not smooth enough, but rather singular - solutions, a main stress should be placed on the respective capabilities to deal with singularities, see some related comments in Rosinger [8, pp. 174-185]. In this regard, let us only mention the following.
  A fundamental property of various spaces of generalized functions which is closely related to their capability to handle a large variety of singularities is that such spaces should have a *flabby sheaf* structure, see for details Kaneko, or Rosinger [10,13]. However, the Schwartz or Sobolev distributions, the Colombeau generalized functions, as well as scores of other frequently used spaces of generalized functions happen to *fail* being flabby sheaves.
  On the other hand, the nowhere dense differential algebras of generalized functions, see (2.22) above, have a structure of flabby sheaves, as shown for instance in Mallios & Rosinger [1].
  Similarly, the far larger class of space-time foam differential algebras of generalized functions dealt with in this paper prove to be flabby sheaves as well, see Rosinger [11] or Mallios & Rosinger [2,3].

- The second aspect is possibly even more controversial. And it is so, especially because of the historical phenomenon that the study of the *linear* theories of generalized functions has from its early modern stages in the 1930s been strongly connected with the then massively emerging theories of linear topological structures.
However, just as with the *nonstandard* reals $^*\mathbb{R}$, so with the various differential algebras of generalized functions, it appears that *infinitesimal* type elements in these algebras play an important role. And the effect is that if one introduces Hausdorff topologies on these algebras, then, when these topologies are restricted to the more regular, smooth, classical type functions, they inevitably lead to the trivial *discrete* topology on them, see related comments in the mentioned places in Rosinger [8], as well as in Remark 1 below.

Compared, however, with the nowhere dense differential algebras of generalized functions, let alone the space-time foam ones dealt with in this paper, the Colombeau algebras suffer from several important limitations. Among them, relevant to this paper is the following.

- There are polynomial type *growth conditions* which the generalized functions must satisfy in the neighbourhood of singularities.

On the other hand, the earlier introduced nowhere dense algebras do not suffer from any of the above two limitations. Indeed, the nowhere dense algebras allow singularities on arbitrary closed nowhere dense sets, therefore, such singularity sets can have arbitrary large positive Lebesgue measure, Oxtoby. Furthermore, in the nowhere dense algebras no any kind of conditions are asked on generalized functions in the neighbourhood of their singularities.
In fact, it is precisely due to the lack of the mentioned type of constraints that the nowhere dense algebras have a flabby sheaf structure, while the Colombeau algebras fail to do so.

Here, for the sake of clarity, let us briefly elaborate on the above. The space $\mathcal{C}^\infty(\mathbb{R}^n)$ is of course a subset of the Colombeau algebra on $\mathbb{R}^n$. However, the smallest flabby sheaf containing $\mathcal{C}^\infty(\mathbb{R}^n)$ is, [34, pp. 143-146]

$$\mathcal{C}^\infty_{nd}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} \exists \;\; \Gamma \subset \mathbb{R}^n, \text{ closed, nowhere dense } : \\ \\ \qquad f|_{\mathbb{R}^n \setminus \Gamma} \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Gamma) \end{array} \right\}$$

and this set of functions is no longer contained in the Colombeau algebra on $\mathbb{R}^n$, since that algebra fails to be a flabby sheaf. More precisely, an arbitrary function $f|_{\mathbb{R}^n \setminus \Gamma} \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Gamma)$, where $\Gamma$, for instance, has a positive Lebesgue measure, can have singularities which the Colombeau algebra on $\mathbb{R}^n$ cannot deal with, since on $\mathbb{R}^n \setminus \Gamma$ and in the neighbourhood of $\Gamma$, such a function $f|_{\mathbb{R}^n \setminus \Gamma} \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Gamma)$ can grow far faster than any polynomial.

In this paper, the use of the *space-time foam* and differential algebras of generalized functions, introduced recently in Rosinger [9,10], brings a further significant enlargement of the possibilities already given by the nowhere dense algebras. Indeed, this time the singularities can be concentrated on *arbitrary* subsets, including *dense* ones, provided that their complementary, that is, the set of nonsingular points, is still dense. Furthermore, as already in the case of the nowhere dense algebras, also in the space-time foam algebras, *no* any sort of condition is asked on the generalized functions in the neighbourhood of their singularities.

One interest in obtaining solutions which are in the space-time foam algebras is that, in view of the interpretations in subsection 1.7., such solutions can be seen as having better *regularity* properties than those obtained earlier in the nowhere dense algebras.

We recall that one could already obtain in the framework of the nowhere dense algebras a very general, and in fact, *type independent* version of the classical Cauchy-Kovalevskaia theorem, see Rosinger [4-8], according to which every analytic nonlinear PDE, together with every non-characteristic analytic initial value problem has a *global* generalized solution, which is *analytic* on the whole domain of definition of the respective PDE, except for a closed nowhere dense set, set which if so desired, can be chosen to have zero Lebesgue measure.
This earlier global and type independent existence results is, fortunately, preserved in the case of the multi-foam algebras as well, and as seen next, it is significantly *strengthened* with respect to the *regularity* properties of the respective global solutions.

Here it can be mentioned that, so far, one could not obtain any kind of similarly general, powerful, and in fact, type independent existence of global solutions result in any of the infinitely many other classes of algebras of generalized functions, including in the Colombeau class of algebras.

And as also mentioned, as far as the Colombeau algebras are concerned, they do not allow even the mere formulation of the global Cauchy-Kovalevskaia theorem, let alone its solution, as obtained in this paper.

### 3.2 The Global Cauchy-Kovalevskaia Theorem

We shall present now in the framework of multi-foam differential algebras of generalized functions the corresponding global version of the Cauchy-Kovalevskaia theorem.

First however, for convenience, let us recall this classical local theorem in its first global formulation, as it was given for the nowhere dense differential algebras of generalized functions, see Rosinger [4-10].

We consider the *general nonlinear analytic* partial differential operator

$$T(x,D)U(x) \ = \ D_t^m U(t,y) - G(t,y,...,D_t^p D_y^q U(t,y),...) \qquad (3.1)$$

where $U : X \longrightarrow \mathbb{C}$ is the unknown function, while $x = (t,y) \in X,\ t \in \mathbb{R},\ y \in \mathbb{R}^{n-1},\ p \in \mathbb{N},\ 0 \leq p < m,\ q \in \mathbb{N}^{n-1},\ p + |q| \leq m$, and $G$ is arbitrary analytic in all of its variables.

Now together with the analytic nonlinear PDE

$$T(x,D)U(x) \ = \ 0, \quad x \in X \qquad (3.2)$$

we consider the non-characteristic analytic hypersurface

$$S \ = \ \{ \ x \ = \ (t,y) \in X \ \mid \ t \ = \ t_0 \ \} \qquad (3.3)$$

for any given $t_0 \in \mathbb{R}$, and on it, we consider the initial value problem

$$D_t^p U(t_0,y) \ = \ g_p(y), \quad 0 \leq p < m, \ \ (t_0,y) \in S \qquad (3.4)$$

Obviously, the analytic nonlinear partial differential operator $T(x,D)$ in (3.1) generates a mapping

$$T(x,D) : \mathcal{C}^\infty(X) \ \longrightarrow \ \mathcal{C}^\infty(X) \qquad (3.5)$$

also, in view of (1.18), (1.19), (2.22) and (3.1), it generates a mapping

$$T(x,D) : A_{nd}(X) \ \longrightarrow \ A_{nd}(X) \qquad (3.6)$$

and the mappings (3.5), (3.6), (1.16) form a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^\infty(X) & \xrightarrow{\quad T(x,D) \quad} & \mathcal{C}^\infty(X) \\ \downarrow & & \downarrow \\ A_{nd}(X) & \xrightarrow[\quad T(x,D) \quad]{} & A_{nd}(X) \end{array} \tag{3.7}$$

In this way, see Rosinger [4-8], we could obtain the earlier global existence result in the nowhere dense differential algebras of generalized functions

**Theorem G C-K**

The analytic nonlinear PDE in (3.2), with the analytic non-characteristic initial value problem (3.3), (3.4) has *global* generalized solutions

$$U \in A_{nd}(X) \tag{3.8}$$

defined on the whole of $X$. These solutions $U$ are *analytic* functions

$$\psi : X \setminus \Sigma \longrightarrow \mathbb{C} \tag{3.9}$$

when restricted to the *open dense* subsets $X \setminus \Sigma$, where the singularity subsets

$$\Sigma \subset X, \quad \Sigma \text{ closed, nowhere dense in } X \tag{3.10}$$

can be suitably chosen. Further, one can choose $\Sigma$ to have zero Lebesgue measure, namely

$$\text{mes } \Sigma = 0 \tag{3.11}$$

□

As a main result of this paper, we shall *strengthen* the above global existence theorem by showing that it also holds in certain classes of multi-foam differential algebras of generalized functions. This will indeed be a strengthening since, as shown next, the respective classes of algebras $A_{Baire\ I}(X)$ and $B_{Baire\ I}(X)$ in (3.12), (3.13) below

- are *surjective* images through algebra homomorphisms of the nowhere dense algebras used in Theorem G C-K, and furthermore, they
- are significantly *smaller* than the nowhere dense algebras.

In this way, since now we shall obtain the existence of global solutions in *smaller* algebras, this result can also be interpreted as a *regularity* result which improves on the earlier existence result in Theorem G C-K, see the respective interpretations in subsection 1.7.

Let us for that purpose return to the two classes of singularities in (1.2) and (1.3), namely $\mathcal{S}_{nd}$ and $\mathcal{S}_{Baire\ I}$, respectively. Further, as in (2.22), let us take for both of them the same right directed partial order, given by $L = (\Lambda, \leq) = \mathbb{N}$.

For simplicity, let us denote by $\mathcal{J}_{nd}(X)$ and $\mathcal{J}_{Baire\ I}(X)$ the respective ideals (1.10) which correspond to these two classes of singularities. Similarly, let us denote by $\mathcal{I}_{nd}(X)$ and $\mathcal{I}_{Baire\ I}(X)$ the respective ideals (2.13) which correspond to the same two classes of singularities.

Then (2.22) gives the nowhere dense algebra $A_{nd}(X)$ both as the multi-foam algebra (1.12) defined by the ideal $\mathcal{J}_{nd}(X)$, as well as the special multi-foam algebra (2.14) defined by the ideal $\mathcal{I}_{nd}(X)$.

Let us now denote by $A_{Baire\ I}(X)$ the special multi-foam algebra which in a similar way is defined by the ideal $\mathcal{I}_{Baire\ I}(X)$.

Further, (1.12) similarly gives the multi-foam algebra $B_{Baire\ I}(X)$ defined by the ideal $\mathcal{J}_{Baire\ I}(X)$.

Now in view of (1.4), (1.10), (2.8) and (2.13), it is clear that

$$\mathcal{I}_{nd}(X) \subset \mathcal{I}_{Baire\ I}(X) \subset \mathcal{J}_{Baire\ I}(X) \tag{3.12}$$

This obviously means that we have the *surjective* algebra homomorphisms

$$\begin{array}{ccccc} A_{nd}(X) & \longrightarrow & A_{Baire\ I}(X) & \longrightarrow & B_{Baire\ I}(X) \\ U = s + \mathcal{I}_{nd}(X) & \longmapsto & U_* = s + \mathcal{I}_{Baire\ I}(X) & \longmapsto & U_{**} = s + \mathcal{J}_{Baire\ I}(X) \end{array} \tag{3.13}$$

which commute with arbitrary partial derivatives, see (1.20), (2.17). And in view of the interpretations in subsection 1.7., we should recall that (3.13) means that the typical generalized functions in $B_{Baire\ I}(X)$ are *more regular* than those both in $A_{nd}(X)$ and $A_{Baire\ I}(X)$.

In this way, as a main result of this paper, we obtain the following *global* Cauchy-Kovalevskaia existence result in algebras with *dense singularities*, a result which also gives *better regularity* properties than those known so far, namely, in the above mentioned earlier Theorem G C-K, see Rosinger [7] :

**Theorem 1.**

The result in Theorem G C-K above holds in any of the following two stronger forms as far as the *regularity* of global solutions is concerned, namely, with

$$U \in A_{Baire\ I}(X) \tag{3.14}$$

or with

$$U \in B_{Baire\ I}(X) \tag{3.15}$$

**Proof.**

In the proof of Theorem G C-K, the global generalized solution $U \in A_{nd}(X)$ is obtained as given by, see Rosinger [4-7]

$$U \;=\; s + \mathcal{I}_{nd}(X) \in A_{nd}(X)$$

where

$$s \;=\; (\psi_\nu \,|\, \nu \in \mathbb{N}) \in (\mathcal{C}^\infty(X))^{\mathbb{N}}$$

and for every compact $K \subset X \setminus \Sigma$ there exists $\nu \in \mathbb{N}$, such that $\psi_\nu = \psi$ on $K$, for $\mu \in \mathbb{N}$, $\mu \geq \nu$.
Now according to the surjective algebra homomorphisms (3.13), we can take the same sequence $s$ and define

$$U = s + \mathcal{I}_{Baire\ I} \in A_{Baire\ I}(X) \tag{3.16}$$

or alternatively

$$U = s + \mathcal{J}_{Baire\ I} \in B_{Baire\ I}(X) \tag{3.17}$$

Then clearly, (3.16), (3.17) will give respectively the global solutions in (3.14) and (3.15).

### 3.3 Connections with Distributions

Let us indicate in short the way the multi-foam algebras can be related to the Schwartz distributions. For that, let us recall the wide ranging and purely algebraic characterization, mentioned at the beginning of this section, of all those differential algebras of generalized functions in which one can embed linearly the Schwartz distributions, a characterization which, as also mentioned, contains the Colombeau algebras as a particular case, see Rosinger [4, pp. 75-88], Rosinger [5, pp. 306-315], Rosinger [6, pp. 234-244].

According to the mentioned characterization, in the case of the multi-foam algebras $B_{L,\mathcal{S}}(X)$, for instance, the necessary and sufficient condition for the existence of such a linear embedding

$$\mathcal{D}'(X) \ \subset \ B_{L,\mathcal{S}}(X) \tag{3.18}$$

is precisely the off diagonality condition (1.15), which as we have seen, does indeed hold. And the linear embedding (3.18) will preserve the differential structure of $\mathcal{C}^\infty(X)$.

In a similar way, in the case of the special multi-foam algebras, the corresponding off diagonality condition (2.18) is again the necessary and sufficient condition for the existence of the linear embedding

$$\mathcal{D}'(X) \subset A_{L,\mathcal{S}_L}(X) \tag{3.19}$$

which again will preserve the differential structure of $\mathcal{C}^\infty(X)$.

### 3.4 Final Remarks

**Remark 1.**

It is important to note that, just like in Mallios & Rosinger [1], where the nowhere dense differential algebras of generalized functions were used, or for that matter in Rosinger [1-11], or Colombeau, Biagioni, Oberguggenberger, Grosser et.al., where other differential algebras of generalized functions appeared as well, so in this paper, where the space-time foam and special space-time foam differential algebras of generalized functions are employed,

there is again *no need* for any topological algebra structure on these algebras.
One of the reasons for the lack of need for any topological algebra structure on the algebras of generalized functions under consideration is the following. It is becoming more and more clear that the classical Kuratowski-Bourbaki topological concept is not suited to the mentioned algebras of generalized functions. Indeed, these algebras prove to contain *nonstandard* type of elements, that is, elements which in a certain sense are infinitely small, or on the contrary, infinitely large. And in such a case, just like in the much simpler case of nonstandard reals $^*\mathbb{R}$, any topology which would be Hausdorff on the whole of the algebras of generalized functions, would by necessity become discrete, therefore trivial, when restricted to usual, standard smooth functions, see for details Biagioni, Rosinger & Van der Walt.

Here, in order to further clarify the issue of the possible limitations of the usual Hausdorff-Kuratowski-Bourbaki concept of topology, let us point out the following. Fundamental results in Measure Theory, predating the mentioned concept of topology, yet having a clear topological nature, have never been given a suitable formulation within that Hausdorff-Kuratowski-Bourbaki concept. Indeed, such is the case, among others, with the Lebesgue dominated convergence theorem, with the Lusin theorem on the approximation of measurable functions by continuous ones, and with the Egorov theorem on the relation between point-wise and uniform convergence of sequences of measurable functions.

Similar limitations of the Hausdorff-Kuratowski-Bourbaki concept of topology appeared in the early 1950s, when attempts were made to turn the convolution of Schwartz distributions into an operation simultaneously continuous in both its arguments. More generally, it is well known that, given a locally convex topological vector space, if we consider the natural bilinear form defined on its Cartesian product with its topological dual, then there will exist a locally convex topology on this Cartesian product which will make the mentioned bilinear form simultaneously continuous in both of its variables, if and only if our original locally convex topology is in fact as particular, as being a normed space topology, see Rosinger & Van der Walt for further details.

It is also well known that in the theory of ordered spaces, and in particular, ordered groups or vector spaces, there are important concepts of convergence, completeness, roundedness, etc., which have never been given a

suitable formulation in terms of the Hausdorff-Kuratowski-Bourbaki concept of topology. In fact, as seen in Oberguggenberger & Rosinger, powerful general results can be obtained about the existence of generalized solutions for very large classes of nonlinear PDEs, by using alone order structures and their Dedekind type order completions, without any recourse to any sort of possibly associated topologies. And the generalized solutions thus obtained can be assimilated with usual measurable functions, or they can be even more regular, such as being Hausdorff-continuous, see Rosinger [12].

Finally, it should be pointed out that, recently, differential calculus was given a new re-foundation by using standard concepts in category theory, such as naturalness. This approach also leads to topological type processes, among them the so called toponomes or $\mathcal{C}$-spaces, which prove to be more general than the usual Hausdorff-Kuratowski-Bourbaki concept of topology, see Nel, and the references cited there.

In this way, we can conclude that Mathematics contains a variety of important *topological type processes* which, so far, could not be formulated in convenient terms using the Hausdorff-Kuratowski-Bourbaki topological concept. And the differential algebras of generalized functions, just as much as the far simpler nonstandard reals $^*\mathbb{R}$, happen to exhibit such a class of topological type processes.

On the other hand, the topological type processes on the nowhere dense differential algebras of generalized functions, used in Mallios & Rosinger [1], for instance, as well as on the space-time foam or special space-time foam differential algebras of generalized functions employed in this paper, see also Mallios [2], Mallios & Rosinger [2,3], can be given a suitable formulation, and correspondingly, treatment, by noting that the mentioned algebras are in fact *reduced powers* see Loš, or Bell & Slomson, of $\mathcal{C}^\infty(X)$, and thus of $\mathcal{C}(X)$ as well. Let us give some further details related to this claim in the case of the space-time foam algebras. The case of the nowhere dense algebras was treated in Mallios & Rosinger [1].

Let us recall, for instance, the definition in (1.12) of the multi-foam algebras, and note that it obviously leads to the relations

$$B_{L,\mathcal{S}}(X) = (\mathcal{C}^\infty(X))^\Lambda/\mathcal{J}_{L,\mathcal{S}}(X) \subseteq (\mathcal{C}(X))^\Lambda/\mathcal{J}_{L,\mathcal{S}}(X) \subseteq \mathcal{C}(\Lambda\times X)/\mathcal{J}_{L,\mathcal{S}}(X) \tag{3.20}$$

assuming in the last term that on $\Lambda$ we consider the discrete topology. A similar situation holds for the special multi-foam algebras $A_{L,\mathcal{S}}(X)$, see (2.14).

Now it is well known, Gillman & Jerison, that the algebra structure of $\mathcal{C}(\Lambda \times X)$ is connected to the topological structure of $\Lambda \times X$, however, this connection is rather sophisticated, as essential aspects of it involve the Stone-Čech compactification $\beta(\Lambda \times X)$ of $\Lambda \times X$. And in order to complicate things, in general $\beta(\Lambda \times X) \neq \beta(\Lambda) \times \beta(X)$, not to mention that $\beta(\Lambda)$ alone, even in the simplest nontrivial case of $\Lambda = \mathbb{N}$, has a highly complex structure.

It follows that a good deal of the discourse, and in particular, the topological type one, in the space-time foam and special space-time foam algebras may be captured by the topology of $\Lambda \times X$, and of course, by the far more involved topology of $\beta(\Lambda \times X)$. Furthermore, the differential properties of these algebras will, in view of (1.17) and (2.9), be reducible termwise to classical differentiation of sequences of smooth functions.

In short, in the case of the mentioned differential algebras of generalized functions, owing to their structure of reduced powers, one obtains a "two-way street" along which, on the one hand, the definitions and operations are applied to sequences of smooth functions, and then reduced termwise to such functions, while on the other hand, all that has to be done in a way which will be compatible with the "reduction" of the "power" by the quotient constructions in (1.12), or in other words, (3.20) and similarly for (2.14). By the way, such a "two-way street" approach has ever since the 1950s been fundamental in the branch of Mathematical Logic, called Model Theory, see Loš. But in order not to become unduly overwhelmed by ideas of Model Theory, let us recall here that the classical Cauchy-Bolzano construction of the real numbers $\mathbb{R}$ is also a reduced power. Not to mention that a similar kind of reduced power construction - in fact, its particular case called "ultra-power" - gives the nonstandard reals ${}^*\mathbb{R}$ as well.

**Remark 2.**

Lately, there has been a growing interest in *noncommutative* studies, and in particular, noncommutative algebras, see Connes. It is therefore appropriate to mention possible connections between such noncommutative methods and the space-time foam and special space-time foam differential algebras of generalized functions in this paper.

In this regard, we recall that, as mentioned at the beginning of this section, in case our constructions start with arbitrary Banach algebra valued, and not merely real or complex valued functions, then the resulting space-time foam and special space-time algebras can still be constructed in the same way, and they will become noncommutative in general.

On the other hand, the emergence of noncommutative studies need not at all mean the loss of interest in, and relevance of commutative structures. Indeed in many problems the commutative approach turns out to be both more effective and also, of course, much more simple.

Finally, it is important to mention here that in the case of singularities of generalized functions, that is, of singularities in a differential context, the approach in Connes falls far short even of the long establish linear theory of Schwartz distributions. Indeed, in such a context, the only differential type operation in Connes, see pp. 19-28, 287-291, is defined as the commutator with a fixed operator. In this way, it is a rather particular derivation, even when considered within Banach algebras. The effect is that, it can only to a small extent deal with the singularities, even when compared with the limited linear Schwartz theory. And certainly, the approach in Connes can deal with even less with singularities on arbitrary closed nowhere dense sets, let alone, on the far larger class of arbitrary dense sets whose complementaries is still dense, such as those in this paper.

# References


[1] Bell J L, Slomson A B : Models and Ultraproducts, An Introduction. North-Holland, Amsterdam, 1969

[2] Biagioni H A : A Nonlinear Theory of Generalized Functions. Lecture Notes in Mathematics, vol. 1421, Springer, New York, 1990

[3] Colombeau J-F : New Generalized Functions and Multiplication of Distributions. Mathematics Studies, vol. 84, North-Holland, Amsterdam, 1984

[4] Connes A : Noncommutative Geometry. Acad. Press, New York, 1994

[5] Finkelstein D : Past-future asymmetry of the gravitational field of a point particle. Physical Review, vol. 110, no. 4, May 1953, 965-967

[6] Geroch R [1] : What is a singularity in General Relativity ? . Annals of Physics, vol. 48, 1968, 526-540

[7] Geroch R [2] : Einstein algebras. Commun. Math. Phys., Springer, vol. 26, 1972, 271-275

[8] Gillman L, Jerison M : Rings of Continuous Functions. Van Nostrand, New York, 1960

[9] Grosser M, Kunzinger M, Oberguggenberger M, Steinbauer R : Geometric Theory of Generalized Functions with Applications to General Relativity. Kluwer, Dordrecht, 2002

[10] Gruszczak J, Heller M : Differential Structure of space-time and its prolongations to singular boundaries. Intern. I. Theor. Physics, vol. 32, no. 4, 1993, 625-648

[11] Heller M [1] : Algebraic foundations of the theory of differential spaces. Demonstratio Math., 24, 1991, 349-364

[12] Heller M [2] : Einstein algebras and general relativity. Intern. J. Theor. Physics, vol. 31, no. 2, 1992, 277-288

[13] Heller M [3] : Thoeretical Foundations of Cosmology, Introduction to the Global Structure of Space-Time. World Scientific, Singapore, London, 1092

[14] Heller M Multarzynski P, Sasin W : The algebraic approach to space-time geometry. Acta Cosmologica, fast. XVI, 1989, 53-85

[15] Heller M, Sasin W [1] : Generalized Friedman's equation and its singularities. Acta Cosmologica, fasc. XIX, 1993: 23-33

[16] Heller M, Sasin W [2] : Sheaves of Einstein algebras. Intern. J. Theor. Physics, vol. 34, no. 3, 1995, 387-398

[17] Heller M, Sasin W [3] : Structured spaces and their application to relativistic physics. J. Math. Phys., 36, 1995, 3644-3662

[18] Kaneko A : Introduction to Hyperfunctions. Kluwer, Dordrecht, 1088

[19] Kirillov A A [1] : Elements of the Theory of Representations. Springer, New York, 1976

[20] Kirillov A A [2] : Geometric Quantization. In (Eds. Arnold V I,Novikov S P) Dynamical Systems IV. Symplectic Geometry and its Application. Springer, New York, 1990, 137-172

[21] Loš J : On the categoricity in power of elementary deductive systems and some related problems. Colloq. Math., 3, 1954, 58-62

[22] Mallios A [1] : Geometry of Vector Sheaves. An Axiomatic Approach to Differential Geometry, vols. I (chapts. 1-5), II (chapts. 6-11) Kluwer, Amsterdam: 1998

[23] Mallios A [2] : Modern Differential Geometry in Gauge Theories. Volume 1 : Maxwell Fields, Volume 2 : Yang-Mills Fields. Birkhauser, Boston, 2006

[24] Mallios A [3] : On an axiomatic treatment of differential geometry via vector.sheaves. Applications. (International Plaza) Math. Japonica, vol. 48, 1998, pp. 93-184

[25] Mallios A [4] : The de Rham-Kähler complex of the Gel'fand sheaf of a topological algebra. J. Math. Anal. Appl., 175, 1993, 143-168

[26] Mallios A [5] : On an abstract form of Weil's intergrality theorem. Note Mat., 12, 1992, 167-202 (invited paper)

[27] Mallios A [6] : On an axiomatic approach to geometric prequantization : A classification scheme á la Kostant-Souriau-Kirillov. J. Mathematical Sciences (former J. Soviet Math.), vol. 98-99

[28] Mallios A, Rosinger E E [1] : Abstract differential geometry, differential algebras of generalized functions, and de Rham cohomology. Acta Applicandae Mathematicae (accepted)

[29] Mallios A, Rosinger E E [2] : Space-time foam dense singularities and de Rham cohomology (to appear)

[30] Mallios A, Rosinger E E [3] : Dense singularities and de Rham cohomology. In (Eds. Strantzalos P, Fragoulopoulou M) Topological Algebras with Applications to Differential Geometry and Mathematical Physics. Proc. Fest-Colloq. in honour of Prof. Anastasios Mallios (16-18 September 1999), pp. 54-71, Dept. Math. Univ. Athens Publishers, Athens, Greece, 2002

[31] Mostow M A : The differentiable space structures of Milnor classifying spaces, simplicial complexes, and geometric realizations. J. Diff. Geom., 14, 1979, 255-293

[32] Nel L D : Differential calculus founded on an isomorphism. Appl. Categorial Structures, 1, 1993, 51-57

[33] Oberguggenberger M B : Multiplication of Distributions and Applications to PDEs. Pitman Research Notes in Mathematics, Vol. 259, Longman, Harlow, 1992

[34] Oberguggenberger M B, Rosinger E E : Solution of Continuous Nonlinear PDEs through Order Completion. Mathematics Studies, vol. 181, North-Holland, Amsterdam, 1994

See also review MR 95k:35002

[35] Oxtoby J C : Measure and Category. Springer, New York, 1971

[36] Rosinger E E [1] : Embedding of the $\mathcal{D}'$ distributions into pseudotopological algebras. Stud. Cerc. Mat., vol. 18, no. 5, 1966, 687-729

[37] Rosinger E E [2] : Pseudotopological spaces, the embedding of the $\mathcal{D}'$ distributions into algebras. Stud. Cerc. Mat., vol. 20, no. 4, 1968, 553-582

[38] Rosinger E E [3] : Distributions and Nonlinear Partial Differential Equations. Lecture Notes in Mathematics, vol. 684, Springer: New York, 1978

[39] Rosinger E E [4] : Nonlinear Partial Differential Equations, Sequential and Weak Solutions. Mathematics Studies, vol. 44, North-Holland, Amsterdam, 1980

[40] Rosinger E E [5] : Generalized Solutions of Nonlinear Partial Differential Equations. Mathematics Studies, vol. 146, North-Holland, Amsterdam, 1987

[41] Rosinger E E [6] : Nonlinear Partial Differential Equations, An A1gebraic View of Geueralized solutions. Mathematics Studies, vol. 164, North-Holland, Amsterdam, 1990

[42] Rosinger E E [7] : Global Version of the Cauchy-Kovalevskaia Theorem for Nonlinear PDEs. Acta Applicandae Mathematicae, Vol. 21, 1990, pp. 331-343

[43] Rosinger E E [8] : Parametric Lie Group Actions on Global Generalized Solutions of Nonlinear PDEs, including a Solution to Hilbert's Fifth Problem, Kluwer Acad. Publ., Dordrecht, Boston, London, 1998

[44] Rosinger E E [9] : Space-time foam differential algebras of generalized functions. Private communication. Vancouver, 1998

[45] Rosinger E E [10] : Dense Singularities and Nonlinear PDEs (to appear)

[46] Rosinger E E [11] : Differential algebras with dense singularities on manifolds. Acta Applicandae Mathematicae, Vol. 95, No. 3, Feb. 2007, 233-256, arXiv:math.DG/0606358

[47] Rosinger E E [12] : Can there be a general nonlinear PDE theory for the existence of solutions ? math.AP/0407026

[48] Rosinger E E [13] : Singularities and flabby sheaves. (to appear)

[49] Rosinger E E [14] : Scattering in highly singular potentials. arXiv:quant-ph/0405172

[50] Rosinger E E [15] : Which are the Maximal Ideals ? arXiv:math.GM/0607082

[51] Rosinger E E, Van der Walt J-H : Beyond topologies ( to appear)

[52] Rosinger E E, Walus Y E [1] : Group invariance of generalized solutions obtained through the algebraic method. Nonlinearity, Vol. 7, 1994, pp. 837-859

[53] Rosinger E E, Walus Y E [2] : Group invariance of global generalized solutions of nonlinear PDEs in nowhere dense algebras. Lie Groups and their Applications, Vol. 1, No. 1, July-August 1994, pp. 216-225

See also reviews : MR 92d:46008, Zbl. Math. 717 35001, MR 92d:46097, Bull. AMS vo1.20, no.1, Jan 1989, 96-101, MR 89g:35001

[54] Sasin W [1] : The de Rham cohomology of differential spaces. Demonstration Mathematica, vol. XXII, no. 1, 1989, 249-270

[55] Sasin W [2] : Differential spaces and singularities in differential space-time. Demonstratio Mathematica, vol. XXIV, no. 3-4, 1991, 601-634

[56] Sikorski R : Introduction to Differential Geometry (in Polish). Polish Scientific Publishers. Warsaw, 1972

[57] Souriau J-M [1] : Structures des Systèmes Dynamiques. Dunod, Paris, 1970

[58] Souriau I-M [2] : Groupes Différentiels. In Differential Geometric Methods in Mathematical Physics. Lecture Notes in Mathematics vol. 836, Springer, New York, 1980, 91-128

[59] Synowiec J A : Some highlights in the development of Algebraic Analysis. Algebraic Analysis and Related Topics. Banach Center Publications, Vol. 53, 2000, 11-46, Polish Academy of Sciences, Warszawa